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LETTER TO THE EDITOR

Extended covariance for the Lagrange equations of motion: a geometric analysis

Carlo Ferrario and Arianna Passerini

Dipartimento di Fisica, Università di Ferrara, Via Paradiso 12, 44100 Ferrara, Italy

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Abstract. The infinitesimal transformations, which leave the Lagrangian structure of the equations of motion unchanged, are intrinsically characterized. A new condition is given with which to obtain alternative Lagrangians.

Classically, the main reason for writing Hamiltonian equations of motion would appear to be the fact that with them one can achieve more freedom whenever covariance is required under transformations of coordinates. The Lagrange equations are covariant under point transformations although relevant properties linked to more general transformations have recently been studied with the renewed interest in the geometrical analysis of Lagrangian mechanics. For example, by taking only those transformations depending also on generalized velocities, it is possible to write a symmetry of the Lagrange function for each constant of motion (see the Noether theorem [1, 2]). Using more general transformations there are more chances of finding alternative Lagrangians [3] and Lagrangian gauge transformations by means of the Dirac theory of constraints [4].

The aim of the present letter is to characterize a class of infinitesimal transformations, called 'Lagrangian', which generalizes the covariance properties of the Lagrange equations. This is done by choosing a different Lagrange function within the 'new' set of coordinates. We will show how these transformations may be canonical or canonoid, Cartan symmetries or simply dynamical symmetries, and how to construct alternative Lagrangians. Here, we are dealing with non-degenerate and time-independent Lagrangians. Moreover, we will limit ourselves to infinitesimal transformations. Thus all necessary and sufficient conditions provided become necessary only whenever finite diffeomorphisms have to be analysed. Such conditions prove to be verified by point transformations and Cartan symmetries which are known to preserve the structure of the Lagrange equations, even at the finite level.

The tools and notations used herein are those used in the geometrical study of classical mechanics (for further details see [7, 8]).

Let Q be the configuration space of the mechanical system and TQ its tangent bundle. As is known [9], one can intrinsically define a one-to-one tensor S on TQ. This is called the vertical endomorphism and endows TQ with the structure of an integrable, almost tangent manifold. Let $\Gamma \in \mathscr{X}(TQ)$ be the second-order vector field associated with Newton's equations of motion and d_S the exterior derivative associated with S, whose action on 0-forms is given by

$$d_S f = df \circ S \qquad f \in \mathscr{F}(TQ). \tag{1}$$

Referring to the notation proposed by Sarlet *et al* [10], use will be made of the set

$$\mathscr{X}_{\Gamma} := \{X \in \mathscr{X}(TQ) \,|\, S[X, \Gamma] = 0\}. \tag{2}$$

These vector fields are known in literature as 'bariation fields of Γ' [11], or 'Newtonoid vector fields' [7], or simply 'non-point transformations' [12]. The infinitesimal generators of point transformations are such that, for any dynamics Γ, they form a subset of $\mathscr{X}_{\Gamma}$ which is closed under Lie brackets. The vector fields of $\mathscr{X}_{\Gamma}$ are important though, in general, they do not have the properties of a lie algebra, since they carry Γ into another second-order vector field.

Nonetheless, not all second-order vector fields give rise to a Lagrangian description: the so-called 'inverse problem in the calculus of variations' is the study of the conditions under which this proves possible. Therefore, once a Lagrange function $\mathscr{L}$ is assigned, one may look for a maximal subset of $\mathscr{X}_{\Gamma}$ whose elements carry Γ into another Lagrangian dynamics.

If $\mathscr{L}$ is non-degenerate

$$\omega_{\mathscr{L}} = -\mathrm{d}\,\mathrm{d}_S\mathscr{L} \tag{3}$$

is a symplectic form. The Lagrange equations are intrinsically written as

$$L_{\Gamma}\,\mathrm{d}_S\mathscr{L} = \mathrm{d}\mathscr{L} \tag{4}$$

or, if one requires explicitly that Γ be a second-order vector field, as

$$i_{\Gamma}\omega_{\mathscr{L}} = \mathrm{d}E_{\mathscr{L}} \tag{5}$$

where

$$e_{\mathscr{L}} := i_{\Gamma}\,\mathrm{d}_S\mathscr{L} - \mathscr{L} \tag{6}$$

is the Lagrangian energy. Equation (5), thought to be an algebraic equation, admits one and only one solution Γ.

Let us examine the action of the infinitesimal transformation generated by $X \in \mathscr{X}_{\Gamma}$ on Γ, $\omega_{\mathscr{L}}$ and $E_{\mathscr{L}}$. Indicating the transformed quantities with Γ', ω' and E', one may ask whether ω' can be derived from a Lagrange function and whether or not Γ is a Lagrangian dynamics for ω'.

A necessary and sufficient condition for the solvability of the inverse problem is given in [13]: all vertical subspaces must be Lagrangian for ω' which must, in turn, be invariant with respect to Γ' and closed (exact for a global result). These last two conditions are identically satisfied under the hypothesis of a transformation of coordinates and, moreover, ω' is globally exact. Thus, the entire problem is reduced to requiring that

$$i_{SZ} i_{SY}(L_X\omega_{\mathscr{L}}) = 0 \qquad \forall\, Y, Z \in \mathscr{X}(TQ). \tag{7}$$

Since ω' is exact, if (7) is satisfied, there are no topological obstructions for the global existence of a Lagrangian, giving rise to the differential equations associated with Γ'.

In fact, since it is easy to verify directly and, as has been observed in [6], once the local form of $X \in \mathscr{X}_{\Gamma}$

$$X = A^i \frac{\partial}{\partial q^i} + (L_{\Gamma}A^i)\frac{\partial}{\partial \dot{q}^i} \tag{8}$$

is given, condition (7) is equivalent to the necessary integrability condition for the equations for $F \in \mathscr{F}(TQ)$

$$\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\frac{\partial A^i}{\partial \dot{q}^j}=\frac{\partial F}{\partial \dot{q}^j} \qquad j=1, n. \tag{9}$$

On the other hand, the fibres are vector space and, therefore, condition (7) is also sufficient to guarantee the existence of F.

The partial differential equation (9) can be found in the literature as a condition for obtaining alternative Lagrangians from dynamical symmetries, i.e. transformations generated by vector fields $X \in \mathscr{X}(TQ)$ such that

$$[X, \Gamma]=0. \tag{10}$$

We will return to this point later since, in the present formulation, it is not necessary for X to satisfy (10). One can rewrite (9) in an intrinsic form as follows: $F \in \mathscr{F}(TQ)$ exists such that

$$(\mathrm{d}_S\mathscr{L}) \circ L_X S+\mathrm{d}_S F=0. \tag{11}$$

This can be verified with a direct calculation recalling that

$$S=\mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i} \tag{12}$$

and then

$$L_X S=-\frac{\partial A^k}{\partial \dot{q}^i}\,\mathrm{d}q^i \otimes \frac{\partial}{\partial q^k}+\left(\frac{\partial A^k}{\partial q^i}-\frac{\partial}{\partial \dot{q}^i}(L_\Gamma A^k)\right)\mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i}+\frac{\partial A^k}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i \otimes \frac{\partial}{\partial \dot{q}^k}. \tag{13}$$

At this point it would be useful to introduce the following definition.

Definition 1. A vector field $X \in \mathscr{X}_\Gamma$ is said to be Lagrangian with respect to a Lagrangian $\mathscr{L}$ admissible for Γ, if a function $\bar{\mathscr{L}} \in \mathscr{F}(TQ)$ exists such that

$$L_X \omega_{\mathscr{L}}=-\mathrm{d}\,\mathrm{d}_S\bar{\mathscr{L}}. \tag{14}$$

The set of such vector fields will be indicated with $\mathscr{X}_\Gamma^{\mathscr{L}} \subset \mathscr{X}_\Gamma$. Since ω' is necessarily exact, condition (11), which is equivalent to (7), is necessary and sufficient for X to belong to $\mathscr{X}_\Gamma^{\mathscr{L}}$.

Proposition. Let Γ be the second-order dynamics associated with the Lagrange function $\mathscr{L}$. Then, $X \in \mathscr{X}_\Gamma$ is a Lagrangian vector field if and only if $F \in \mathscr{F}(TQ)$ exists such that

$$(\mathrm{d}_S\mathscr{L}) \circ L_X S+\mathrm{d}_S F=0 \tag{15}$$

and, in such case, one finds

$$L_X \omega_{\mathscr{L}}=-\mathrm{d}\,\mathrm{d}_S(L_X\mathscr{L}-L_\Gamma F). \tag{16}$$

Proof. Let us define the following functions

$$G=i_X\,\mathrm{d}_S\mathscr{L}-F \tag{17}$$

$$\bar{\mathscr{L}}=L_X\mathscr{L}-L_\Gamma F. \tag{18}$$

By differentiating (17) with respect to Γ, and by taking both (2) and (4) into account, one obtains

$$\bar{\mathscr{L}} = L_\Gamma G \tag{19}$$

which, along with the relationship

$$L_\Gamma(\mathrm{d}G - i_X\omega_{\mathscr{L}}) = \mathrm{d}L_\Gamma\, G - i_{[\Gamma,X]}\omega_{\mathscr{L}} \tag{20}$$

provides the following:

$$i_{[X,\Gamma]}\omega_{\mathscr{L}} = L_\Gamma(\mathrm{d}G - i_X\omega_{\mathscr{L}}) - \mathrm{d}\bar{\mathscr{L}}. \tag{21}$$

The exterior derivative of (17) implies

$$L_X\, \mathrm{d}_S\mathscr{L} - \mathrm{d}F = \mathrm{d}G - i_X\omega_{\mathscr{L}}. \tag{22}$$

If hypothesis (15) holds true, the left-hand side of (22) is a semibasic 1-form. In fact, keeping in mind that

$$L_XS \circ S = -S \circ L_XS \tag{23}$$

one has

$$(L_X\, \mathrm{d}_S\mathscr{L} - \mathrm{d}F) \circ S = -(\mathrm{d}_S\mathscr{L}) \circ L_XS - \mathrm{d}_SF = 0. \tag{24}$$

Thus

$$i_{SY}(\mathrm{d}G - i_X\omega_{\mathscr{L}}) = 0 \qquad \forall Y \in \mathscr{X}(TQ) \tag{25}$$

and, moreover, since

$$S[SY, \Gamma] = SY \tag{26}$$

one also has

$$i_{[SY,\Gamma]}(\mathrm{d}G - i_X\omega_{\mathscr{L}}) = i_Y(\mathrm{d}G - i_X\omega_{\mathscr{L}}) \qquad \forall Y \in \mathscr{X}(TQ). \tag{27}$$

Finally, by contracting (21) with respect to SY and using (27), it follows that

$$i_Y(\mathrm{d}G - i_X\omega_{\mathscr{L}}) = L_{SY}\bar{\mathscr{L}} \qquad \forall Y \in \mathscr{X}(TQ). \tag{28}$$

Therefore,

$$\mathrm{d}G - i_X\omega_{\mathscr{L}} = \mathrm{d}_S\bar{\mathscr{L}} \tag{29}$$

and

$$L_X\omega_{\mathscr{L}} = -\mathrm{d}\,\mathrm{d}_S\mathscr{L}. \tag{30}$$

The converse is trivial: whenever (30) holds true (7) is satisfied. □

In the light of these simple calculations some interesting comments may be made. First of all, looking at (11), it becomes evident that F is defined up to an arbitrary function $a \in \mathscr{F}(Q)$ and that G is defined as well. Then $\bar{\mathscr{L}}$ is defined, except for the so-called gauge function $L_\Gamma a$. This indetermination is trivial and will be neglected from this point on. For example, looking at (30), we can say that X generates a canonical transformation (see [14]) if and only if $\bar{\mathscr{L}}$ does not depend on the velocities, thus omitting and gauge function.

On the other hand, substituting (29) in (21) one gets

$$i_{[X,\Gamma]}\omega_{\mathscr{L}} = L_\Gamma\, \mathrm{d}_S\bar{\mathscr{L}} - \mathrm{d}\bar{\mathscr{L}} \tag{31}$$

whose right-hand side, compared with (4), shows that $\bar{\mathscr{L}}$ is an alternative Lagrangian for Γ if and only if X is a symmetry of Γ. However, it is possible to ease up this condition by choosing the alternative Lagrangian in a different way, as will now be demonstrated.

In order to do so we must recall the definition of canonoid vector fields [14].

Definition 2. A vector field $Z \in \mathscr{X}(TQ)$ is said to be canonoid with respect to Γ if

$$i_\Gamma \omega_{\mathscr{L}} = \mathrm{d}E_{\mathscr{L}} \tag{32}$$

involves

$$i_\Gamma L_Z \omega_{\mathscr{L}} = \mathrm{d}C \qquad C \in \mathscr{F}(TQ) \tag{33}$$

i.e. is bi-Hamiltonian.

If Z is canonoid with respect to any Hamiltonian vector field (the term 'Hamiltonian' being referred to the symplectic form $\omega_{\mathscr{L}}$) it generates a canonical transformation; that is to say

$$L_Z \omega_{\mathscr{L}} = 0. \tag{34}$$

If $X \in \mathscr{X}_\Gamma$ is a canonical transformation, condition (7) is trivially satisfied and $X \in \mathscr{X}_\Gamma^{\mathscr{L}}$. One immediately sees that

$$i_{[X,\Gamma]} \omega_{\mathscr{L}} = \mathrm{d} L_X E_{\mathscr{L}} \tag{35}$$

and, therefore, as the left-hand side of (35) is semibasic because of $S[X, \Gamma] = 0$, one finds

$$\mathrm{d}_S L_X E_{\mathscr{L}} = 0. \tag{36}$$

Since the right-hand side of (29) is zero, contracting with respect to Γ and using (5) we also find the useful relationship

$$\bar{\mathscr{L}} = L_\Gamma G = -L_X E_{\mathscr{L}}. \tag{37}$$

Conversely, if X is the generator of a canonical transformation and satisfies (36), then X belongs to $\mathscr{X}_\Gamma$. By requiring less particular features, we may study canonoid transformations in TQ.

Let us suppose that $X \in \mathscr{X}_\Gamma$ is canonoid with respect to Γ: from (33) it follows that $[X, \Gamma]$ is (globally) Hamiltonian. In fact,

$$i_{[X,\Gamma]} \omega_{\mathscr{L}} = \mathrm{d}g \tag{38}$$

with $g = L_X E_{\mathscr{L}} - C$. Since $S[X, \Gamma] = 0$, one obtains

$$0 = (i_{[X,\Gamma]} \omega_{\mathscr{L}}) \circ S = \mathrm{d}_S g. \tag{39}$$

Moreover, if $X \in \mathscr{X}_\Gamma^{\mathscr{L}}$, from (31) one gets

$$L_\Gamma(\mathrm{d}_S \tilde{\mathscr{L}}) - \mathrm{d}\tilde{\mathscr{L}} = 0 \tag{40}$$

where $\tilde{\mathscr{L}} = \bar{\mathscr{L}} + g$ proves to be an alternative Lagrangian. Any symmetry X of the dynamics Γ is canonoid (with $C = L_X E$) although the converse is not true. Therefore, together equations (15) and (33) constitute a less restrictive condition than the one proposed in [5] to guarantee the existence of alternative Lagrangians.

The alternative Lagrangians obtained from canonoid transformations are not necessarily trivial, as may be seen from the following example.

Let $Q=\mathbb{R}^2$ and let

$$\Gamma=\dot{q}^i\frac{\partial}{\partial q^i}-q^i\frac{\partial}{\partial \dot{q}^i} \tag{41}$$

be the dynamics of the bidimensional isotropic harmonic oscillator. One has, as an admissible Lagrangian

$$\mathscr{L}=\tfrac{1}{2}(\dot{q}^i\dot{q}^i-q^iq^i). \tag{42}$$

Let us construct the infinitesimal transformation given in (8) with

$$A^1=(\dot{q}^1)^2+\dot{q}^2+\tfrac{1}{2}(q^1)^2 \tag{43a}$$

$$A^2=(\dot{q}^2)^3+\dot{q}^1+(q^2)^2\dot{q}^2+q^2. \tag{43b}$$

X is a Lagrangian vector field, with

$$F=\tfrac{2}{3}(\dot{q}^1)^3+\tfrac{3}{4}(\dot{q}^2)^4+\dot{q}^1\dot{q}^2+\tfrac{1}{2}(q^2)^2(\dot{q}^2)^2 \tag{44}$$

satisfying (15), therefore

$$i_{[X,\Gamma]}\omega_{\mathscr{L}}=\mathrm{d}(\tfrac{1}{2}(q^1)^3). \tag{45}$$

X is also a canonoid vector field and gives rise to the alternative Lagrangian

$$\tilde{\mathscr{L}}=(\dot{q}^2)^2-(q^2)^2-L_\Gamma(q^1q^2+\tfrac{1}{4}(q^2)^4) \tag{46}$$

which is not trivial since the difference between $\mathscr{L}$ and $\tilde{\mathscr{L}}$ is not a gauge function $L_\Gamma f$, with $f\in\mathscr{F}(Q)$: they are not equivalent Lagrangians (see [9]).

Finally, we can analyse the properties of the Cartan symmetries, the definition of which, as given in [1] and in [5], proves equivalent to the following: a function $F\in\mathscr{F}(TQ)$ exists such that

$$(\mathrm{d}_S\mathscr{L})\circ L_XS+\mathrm{d}_SF=0 \tag{47a}$$

$$L_X\mathscr{L}=L_\Gamma F. \tag{47b}$$

or simply: $X\in\mathscr{X}_\Gamma^{\mathscr{L}}$ is such that

$$\bar{\mathscr{L}}=0. \tag{48}$$

In such a case the proof of the proposition is just the proof of the Noether theorem: (19), (31) and (29) respectively, ensure that G is a constant of motion, X is a dynamical symmetry and, moreover, generates a canonical transformation.

Now, some observations must be made regarding the transformed equations of motion. From (18) it is easy to understand that the admissible Lagrangian for Γ' is not $\mathscr{L}'$ but $\mathscr{L}'-L_\Gamma F$. If we take a passive point of view we can see that, if we write the same equations of motion into the new set of coordinates, they need a different Lagrangian. When we are dealing with a Cartan symmetry, the equations are more than covariant: they are invariant and the new Lagrangian has the same functional form as the old one. Nonetheless, even in this case, the new Lagrangian is different once evaluated in the same point of the space.

One gets the same Lagrange function by taking point transformations, since they are always Lagrangian with $F=0$. The consequences of a Lagrangian transformation in the phase space T^*Q are simple to analyse. Whenever the Lagrange function is not invariant one must, in general, modify the Legendre mapping by adding the fibre derivative [7] $F(\bar{\mathscr{L}})$. The modified mapping carries the new Lagrange equations into new Hamilton equations. The latter are linked to the old ones by means of the canonical transformation generated by the Hamiltonian field $X_G \in \mathscr{X}(T^*Q)$ such that

$$i_{X_G}(F(\mathscr{L})_* \omega_{\mathscr{L}}) = \mathrm{d}F(\mathscr{L})_* G \tag{49}$$

where $F(\mathscr{L})_*$ is the push-forward of the Legendre mapping. From (19) we can see that the variation of the Hamiltonian function $H = F(\ell)_* E_{\mathscr{L}}$ under said canonical transformation is the opposite of $\bar{\mathscr{L}}$. In fact

$$\bar{\mathscr{L}} = -F(\mathscr{L})_* \{H, F(\mathscr{L})_* G\}. \tag{50}$$

In the particular case in which $X \in \mathscr{X}_\Gamma^{\mathscr{L}}$ is canonical in TQ, since $\bar{\mathscr{L}} \in \mathscr{F}(Q)$, the Legendre mapping does not change and

$$X_G = F(\mathscr{L})_* X. \tag{51}$$

The set of infinitesimal generators maintaining the Lagrangian structure of the equations of motion has been defined on TQ; these are called 'Lagrangian transformations'. It has been underlined that point transformations and Cartan symmetries are simply subsets of this set, at least at the infinitesimal level. The geometrical features of the Lagrangian transformations have been demonstrated, greater depth being given to their relationships with the inverse problem and with the problem of alternative Lagrangian descriptions. Moreover, emphasis has been given to a one-to-one correspondence between Lagrangian transformations on TQ and canonical transformations on T^*Q.

The analysis brought to light in the present letter constitutes a useful tool for understanding the conditions under which the vector fields of $\mathscr{X}_\Gamma^{\mathscr{L}}$ generate one-parameter groups; that is to say their properties are maintained in finite transformations as well.

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